CHAP: 1: ROTATIONAL DYNAMICS

* **Circular Motion**: It is the motion of a body along the circumferen of circle

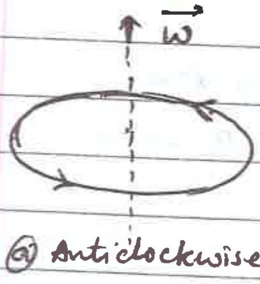
- **Characteristics of circular motion:**
  1) It is an accelerated motion; as the direction of velocity changes at every instant.
  2) It is a periodic motion; as the body repeats its path along the same trajectory (in same time)
- **Kinematics of circular Motion**

In translational motion the quantities displacement $\vec{s}$, velocity $\vec{v} = \frac{d\vec{s}}{dt}$ & acceleration $\vec{a} = \frac{d\vec{v}}{dt}$ are analogous in circular motion as angular displacement $\vec{\theta}$, angular velocity $\vec{\omega} = \frac{d\vec{\theta}}{dt}$ & angular acceleration $\vec{\alpha} = \frac{d\vec{\omega}}{dt}$ respectively.

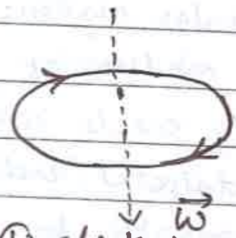
	Translational motion		circular motion (angular)	
1) Displacement	$\vec{s}$	metre,	$\vec{\theta}$	radian
2) velocity	$\vec{v} = \frac{d\vec{s}}{dt}$	m/s,	$\vec{\omega} = \frac{d\vec{\theta}}{dt}$	rad/s
3) Acceleration	$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2s}{dt^2}$	m/s^2,	$\vec{\alpha} = \frac{d\vec{\omega}}{dt} = \frac{d^2\theta}{dt^2}$.	rad/s^2.

The relation bet^n linear and angular velocity is $\vec{v} = \vec{\omega} \times \vec{r}$ --- ($\vec{r}$ is radius vector of position vector), or $v = \omega r$.

The direction of $\vec{\omega}$ is always along the axis of rotation, given by right hand thumb rule i.e curled fingers of right hand along the sense of rotation & the outstreched thumb gives the direction of $\vec{\omega}$.



(a) Anticlockwise rotation.



(b) clockwise rotation

Also ω is related to period (Periodic time) T & freq^n n as

$$\omega = 2\pi n = \frac{2\pi}{T}.$$

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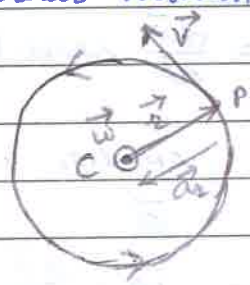
* Uniform Circular Motion :-

It is the motion of a body along the circumference of circle with constant speed. (constant angular velocity).

In this case the direction of velocity changes at every instant with constant magnitude & it is always along the tangential to the path. The accⁿ responsible for this is centripetal or radial accⁿ.

$\vec{a}_r = -\omega^2 \vec{r}$ (-ve sign is due to $\vec{a}$ & $\vec{r}$ are opposite)

It's magnitude is $a = \omega^2 r = \frac{V^2}{r} = V\omega$. This accⁿ is always directed towards the centre of circular motion.



$\vec{r}$ = Radius vector
$\vec{V}$ = linear velocity
$\vec{\omega}$ = Angular velocity
$\vec{a}_r$ = Centripetal (radial) acceleration.

The linear acceleration is

$\vec{a} = \vec{a}_r + \vec{a}_T$ —— ① $\vec{a}_r$ = Radial accⁿ (component)

where $\vec{a}_T$ = Tangential accⁿ (component)

$\vec{a}_r = \vec{\omega} \times \vec{V}$

$\vec{a}_T = \vec{\alpha} \times \vec{r}$ --- but $\alpha = \frac{d\omega}{dt} = 0$ --- ω = constant

$\therefore a_T = 0$

$\therefore$ For Uniform circular motion $\underline{\vec{a} = \vec{a}_r}$

but magnitude of $\vec{a}$ is $a = \sqrt{a_r^2 + a_T^2}$.

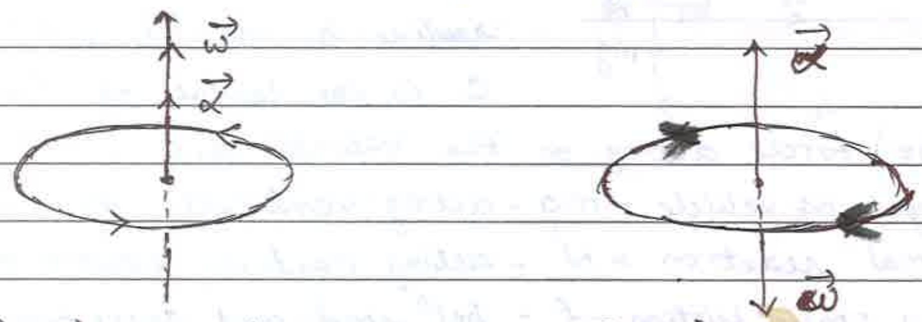
* Non uniform Circular Motion :-

It is a motion of a body along the circumference of circle with changing its speed.

Hear centripetal (radial) accⁿ $\vec{a}_r$ is not constant. but the accⁿ responsible for changing the magnitude of velocity is directed along or opposite to the velocity, so it is tangential accⁿ $\vec{a}_T$.

The tangential velocity ($\vec{V}$), as well as angular velocity $\vec{\omega}$ are changing at every instant. $\therefore \vec{\alpha} = \frac{d\vec{\omega}}{dt}$. The plane of particles remains same.

The direction of $\vec{\alpha}$ is along the axis of rotation; it is in the direction or in opposite direction of $\vec{\omega}$.



ⓐ $\vec{\omega}$ increasing ⓑ $\vec{\omega}$ decreasing

* <u>Centripetal force</u> (CPF ⇒ F_{cp}):

It is the force acting on a body directed towards the centre along the radius of circle, when it performes circular motion.

The centripetal (radial) acc^n $\left[\vec{a_r} = -\omega^2 \vec{r}\right]$ is responsible for circular motion; the force providing this acc^n is centripetal force.

$$\therefore \vec{F_{cp}} = -m\omega^2 \vec{r} = -\frac{mv^2}{r}\hat{r}$$

The magnitude of force is $F_{cp} = mr\omega^2 = \frac{mv^2}{r}$.

* <u>Centrifugal force</u> (C.F.F. ⇒ F_{CF}):

It is the force acting on a body directed away from the centre, along the radius, when it performs circular motion.

The centripetal (radial) acc^n $\left[\vec{a_r} = -\omega^2 \vec{r}\right]$ is responsible for centrifugal force in opposite direction.

$$\therefore \vec{F_{CF}} = +m\omega^2 \vec{r} = +\frac{mv^2}{r}\hat{r}.$$

Its magnitude is $F_{CF} = mr\omega^2 = \frac{mv^2}{r}$.

It is a non-real force (Pseduo force) but not an imaginary force.

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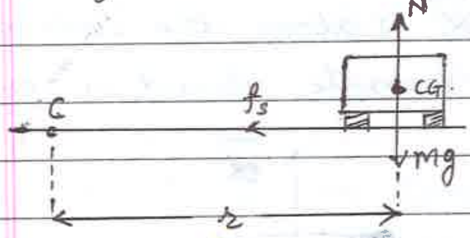
* Applications of Uniform Circular Motion.

A] Find the equation of maximum possible speed of a vehicle along a horizontal Circular track.

$\Longrightarrow$



Consider a vehicle (car) of mass m, on a horizontal circular track of radius r, moving with velocity v. 'C' is the centre of the track.

The forces acting on the vehicle are

i) weight of vehicle = mg - acting vertically downwards.

ii) Normal reaction = N - acting vertically upward balances mg

iii) Force of static friction = f_s - bet^n road and tyres towads centre balances with centripetal force

$$\therefore N = mg \quad — ①$$

$$f_s = \frac{mv^2}{r} = mr\omega^2 \quad ②$$

Divide ② by ①

$$\frac{f_s}{N} = \frac{r\omega^2}{g} = \frac{v^2}{rg} \quad — ③$$

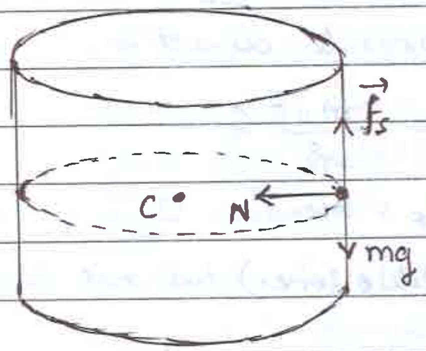
As N, r & g are constant, ∴ f_s increases with increase in

$$\therefore \frac{(f_s)_{max}}{N} = \mu_s = \frac{v^2_{max}}{rg}$$

$$\therefore \boxed{v_{max} = \sqrt{\mu_s r g}} \quad — ④$$

B] Find the equation of minimum possible speed of a vehicle in a well of depth (wall).

$\Longrightarrow$



Well of death is a vertical cylindrical wall of radius r. inside which a vehicle of mass m is driven in horizontal circles with speed v.

The forces acting on vehicle are

i) Normal reaction = N - acting horizontally towards centre & balance with centripetal force

ii) weight of vehicle = mg - acting vertically downwards.

iii) Force of static friction = f_s - acting vertically upward betⁿ vertical wall & tyres, balances with weight mg.

$$\therefore N = \frac{mv^2}{r} = mr\omega^2 \quad \text{--- (1)}$$

$$f_s = mg \quad \text{--- (2)}$$

The force of static friction f_s is always less than or equal to $\mu_s N$.

$$\therefore f_s \le \mu_s N$$

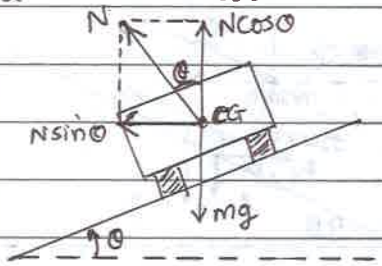
$$\therefore mg \le \mu_s \left(\frac{mv^2}{r}\right)$$

$$\therefore g \le \frac{\mu_s v^2}{r}$$

$$\therefore v^2 \ge \frac{rg}{\mu_s}$$

$$\therefore \boxed{V_{min} = \sqrt{\frac{rg}{\mu_s}}} \quad \text{--- (3)}$$

c] Obtain an expressions of (a) most safe speed (b) angle of banking and hence find minimum & maximum speed of vehicle on a banked road.

⟹ 

C.G. = Centre of gravity
mg = Gravitational force (weight)
N = Normal reaction
θ = Angle of banking.

Consider a vehicle of mass 'm' moving along a banked curve road ~~of~~ with speed V, 'r' is the radius of curvature of banked road with angle of banking θ with horizontal. The vehicle to be a point at which ignoring friction the forces acting on vehicle are

1) weight (mg) - vertically downward
2) Normal reaction (N) - perpendicular to the road surface.

As the motion of the vehicle is along a horizontal circle, the resultent force must be horizontal directed towards the centre of the track.

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ie vertical force mg must be balanced.

So vertical component $N\cos\theta$ balances with mg. & horizontal component ~~μ~~ $N\sin\theta$ provides necessary centripetal force (mv^2/r).

$\therefore N\cos\theta = mg$ — (1)

$N\sin\theta = \frac{mv^2}{r}$ — (2)

Divide (2) by (1)

$\therefore \tan\theta = \frac{v^2}{rg}$ — (3)

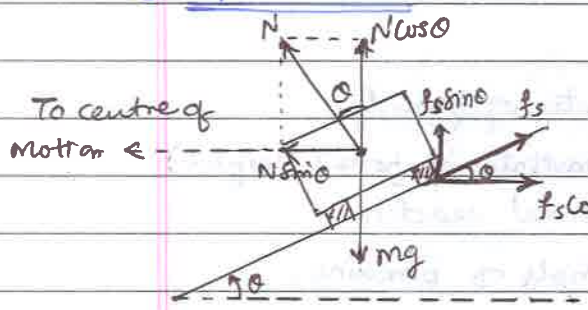
(a) Most safe speed: For a particular road r & θ are fixed ∴ from eqn (3) the most safe speed is

$$V_s = \sqrt{rg\tan\theta} \quad — (4)$$

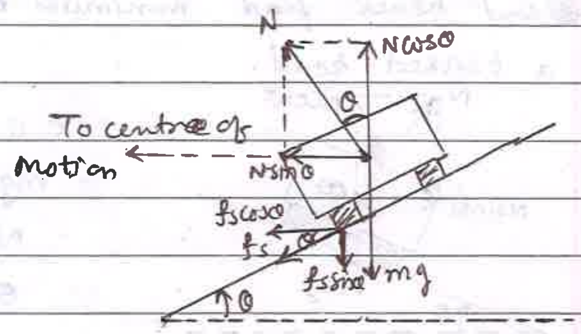
(b) Angle of banking:- while designing the road the angle of banking from eqn (3) is

$$\theta = \tan^{-1}\left(\frac{v^2}{rg}\right) \quad — (5)$$

(c) Speed limit:-



(a) Banked road - Lower speed limit



(b) Banked road - Upper speed limit.

Figs (a) & (b) show vertical section of a vehicle on a rough curve road. of radius 'r' & angle of banking θ-

If the vehicle is running exactly at speed $V_s = \sqrt{rg\tan\theta}$, the forces acting on vehicle as shown in fig. (mg & N), but in practice the speed of vehicle is different because the component of force of static friction betn road & tyres.

1) From fig (a) if the speed is $v_1 < \sqrt{rg\tan\theta}$,

$\therefore \frac{mv_1^2}{r} < N\sin\theta$. (or $N\sin\theta$ is greater than centrifugal force $\frac{mv_1^2}{r}$)

In this case the direction of f_s is directed along the inclination of road (upward) $\therefore$ its horizontal component ($f_s\cos\theta$) is parallel & opposite to $N\sin\theta$.

$\therefore$ From fig (a)

$$mg = f_s\sin\theta + N\cos\theta \quad \text{— (1)}$$

$$\frac{mv_1^2}{r} = N\sin\theta - f_s\cos\theta \quad \text{— (2)}$$

For minimum possible speed $f_s = \mu_s N$ $\therefore$ From eq^ns (1) & (2)

$$\boxed{(v_1)_{min} = V_{min} = \sqrt{rg\left(\frac{\tan\theta - \mu_s}{1 + \mu_s\tan\theta}\right)}} \quad \text{— (3)}$$

For $\mu_s \geq \tan\theta$, $V_{min} = 0$. It is true for most of rough roads, banked at smaller angles.

2) From fig (b) if the speed is $V_2 > \sqrt{rg\tan\theta}$,

$\therefore \frac{mV_2^2}{r} > N\sin\theta$. (or $N\sin\theta$ is less than the centrifugal force $\frac{mV_2^2}{r}$).

In this case the direction of f_s is directed along the inclination of road (downwards), its horizontal component ($f_s\cos\theta$) is parallel & in the direction of $N\sin\theta$.

$\therefore$ From fig (b)

$$mg = N\cos\theta - f_s\sin\theta \quad \text{— (4)}$$

$$\frac{mV_2^2}{r} = N\sin\theta + f_s\cos\theta \quad \text{— (5)}$$

for max^m possible speed, f_s is max^m $\therefore f_s = \mu_s N$ $\therefore$ From eq^n (4) & (5)

$$\boxed{(V_2)_{max} = V_{max} = \sqrt{rg\left(\frac{\tan\theta + \mu_s}{1 - \mu_s\tan\theta}\right)}} \quad \text{— (6)}$$

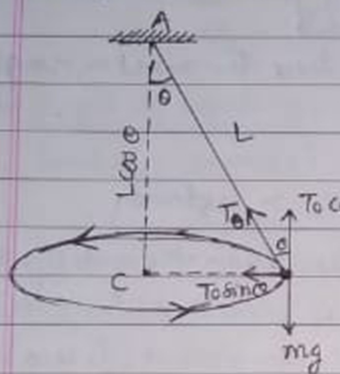
If $\mu_s = \cot\theta$, $V_{max} = \infty$, but $(\mu_s)_{max} = 1$. Thus for $\theta = 45°$, $V_{max} = \infty$; For heavily banked road, min^m limit may be important.

For $\mu_s = 0$, From eq^ns (3) & (6) $v = \sqrt{rg\tan\theta}$. which is the safest speed on a banked road, as we don't take the help of friction.

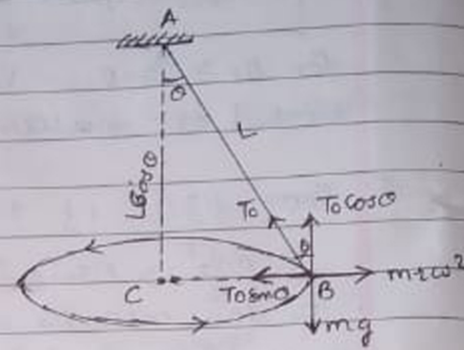
2) Define simple pendulum and conical pendulum. Draw its neat diagrams & obtain an expression for its period & freq^n.

⟹ <u>Simple pendulum</u>:- A point mass (bob) connected to a long, flexible, massless & inextensible string suspended from a rigid support & string is made to oscillate in single vertical plane, is called as simple pendulum.

<u>Conical pendulum</u>: A point mass (bob) connected to a long, flexible, massless & inextensible string suspended from a rigid support & string <u>moves</u> along the surface of a right circular cone of vertical axis & a point mass performs a uniform horizontal circular motion, is called as conical pendulum.



(a) In an Inertial frame



(b) In a non Inertial frame.

A conical pendulum having point mass (bob) of mass m & string of length L. At position B the forces acting on pendulum are

1) The weight (mg) - directed vertically downward.
2) The tension (T_0) in the string - directed along the string towards the support.

As the motion of a bob is a horizontal circular motion, the resultant force must be horizonal & directed towards centre C & vertical forces are balanced; cancel to each other. So force T_0 is resolved as $T_0\cos\theta$ - vertical component balance with mg & horizonal component $T_0\sin\theta$ becomes centripetal force.

$\therefore T_0 \sin\theta = \left(\frac{mv^2}{r}\right) = mr\omega^2$ —— ①

$T_0 \cos\theta = mg$ —— ②

Divide ① by ②

$\therefore \omega^2 = \frac{g\sin\theta}{r\cos\theta}$ - - - $(r = L\sin\theta)$

$\therefore \omega = \frac{2\pi}{T} = \sqrt{\frac{g}{L\cos\theta}}$

$\therefore \boxed{T = 2\pi\sqrt{\frac{L\cos\theta}{g}}}$ —— ③ → Period of conical pendulum.

The freq^n of conical pendulum is $n = \frac{1}{T}$.

$\boxed{n = \frac{1}{2\pi}\sqrt{\frac{g}{L\cos\theta}}}$ —— ④

* Vertical circular Motion :-

There are two types of vertical circular motions.

a) A controlled vertical circular motion in which speed is either kept constant or not totally controlled by gravity.
eq. Giant wheel

b) A Controlled vertical circular motion which is controlled only by gravity. (There is interconversion of KE & gravitational P.E).
eq. A body tied to a string.

Q. Describe vertical circular motion of a point mass under gravity. State the eq^{ns} of tensions & linear velocities at diff. positions.

⟹ Case I] Mass tied to a string :-

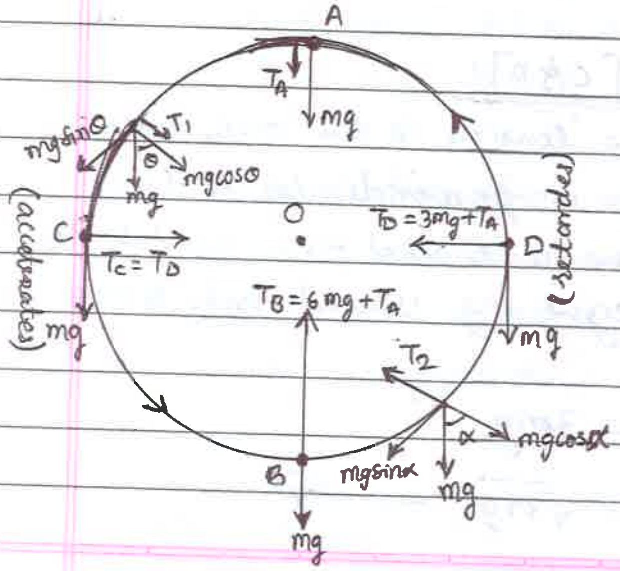


Fig. shows a bob of mass m tied to a string of length 'l' which is massless, inextensible & flexible. It is whirled along a vertical circle of radius $r = l$. At any position of a bob there are only two forces acting on the bob.

1) weight (mg), vertically downward, which is constant.

2) Tension (T) in the string & towards centre. It's magnitude changes periodically with time & location.

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Uppermost (Top) position [A]:

The weight mg & tension T_A are downwards, their resultant is centripetal force, As V_A is the speed of bob at A.

$\therefore \; T_A + mg = \frac{mV_A^2}{r}$ — ①

For minimum possible speed, $T_A = 0$

$\therefore (V_A)_{min} = \sqrt{rg}$ — ②

Lowest (bottom) position [B]:

At B the tension T_B is vertically upward & opposite to mg. As V_B is the speed of bob at B, the resultant force is

$T_B - mg = \frac{mV_B^2}{r}$ — ③

when bob coming down from uppermost to lowermost point due to gravity only. The decrease in gravitational PE is converted into change in KE.

$\therefore (P.E)_A = (KE)_B - (KE)_A$

$mg \times 2r = \frac{1}{2}mV_B^2 - \frac{1}{2}mV_A^2$

$\therefore V_B^2 - V_A^2 = 4rg$ — ④

From eq^n ② & ④

$(V_B)_{min} = \sqrt{5rg}$ — ⑤

From eq^n ① & ③

$T_B - T_A = 6mg$ — ⑥

Horizontal position (midway) [C & D]:

Force due to the tension is the only force towards the centre as weight mg is perpendicular to tension

$\therefore$ Force due to tension is the centripetal force used to change direction of velocity & weight mg is used only to change the speed.

$\therefore T_C - T_A = T_D - T_A = 3mg.$

$\therefore (V_C)_{min} = (V_D)_{min} = \sqrt{3rg}$ — ⑦

Arbitrary position (Any point):

Force due to the tension & weight are neither along the same line, nor perpendicular. So Tangential component of mg is used to change the speed. It decreases while going up & increases while going down.

Case II] point mass tied to a rod:

Consider a bob tied to a massless & rigid rod, whirled along a vertical circle. The diff. betn rod & string is that the string needs some tension at all points, so the certain minm speed ($V_A = \sqrt{rg}$) is necessary at the uppermost point in the string, but for rod, as it is rigid this condition of V_{min} is not necessary. So zero speed is possible at the uppermost point.

$$\therefore (V_{lowermost})_{min} = V_B = \sqrt{4rg} = 2\sqrt{rg}.$$

V_{min} at the rod horizontal position $= \sqrt{2rg}$.

& $T_{lowermost} - T_{uppermost} = 6mg$.

* Sphere of Death: This is popular show in a circus. During this, two wheeler rider undergo rounds inside a hollow sphere, starting with small horizontal circles, then they performs revolutions along vertical circles. This vertical motion is same as a point mass tied to a string, except that the force due to tension T is replaced by the normal reaction N.

Initially there are nearly horizontal circles. The linear speed is more for larger circles., but angular speed is more for smaller circles (at starting or stopping). according to the theory of conical pendulum.

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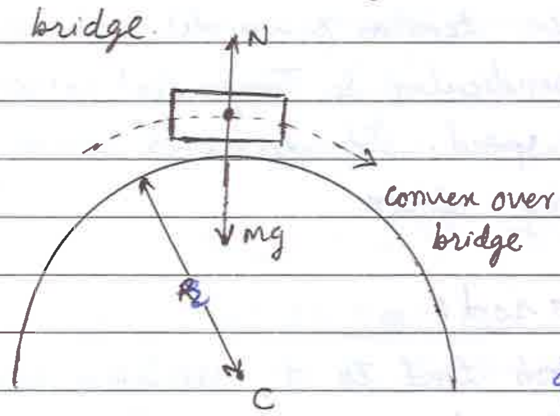
Q. Find the speed of vehicle at the top of a convex over bridge.

⟹



A vehicle of mass m, moving at the top of a convex over bridge of radius of curvature 'r' with speed v. The forces acting on the vehicle are
a) weight mg - vertically downward.
b) Normal reaction - N - vertically upward

The resultent of these two forces provides necessary centripet force;

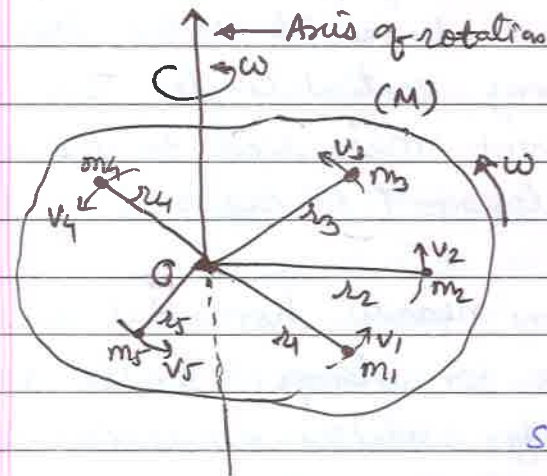
$\therefore \quad mg - N = \frac{mv^2}{r}$ —— ①

As speed increases, N goes on decreasing. Normal reaction is the indication of contact. for just maintaining contact. N = 0, so the upper limit on the speed is

$\boxed{v_{max} = \sqrt{rg}}$ —— ②

Q. Obtain an expression of rotational KE of a rigid body & hence show moment of inertia as an analogous quantity for mass & define it. (M.I.)

⟹



A rigid body of mass M is rotating about an axis passing through it; with constant angular velocity ω. The body consist of 'n' no. of particles of masses m_1, m_2, m_3 -- m situated at perpendicular distance r_1, r_2, r_3 -- r_n respectively from the axis of rotation.

All particles performs UCM with same angular velocity 'ω' but their linear velocities are different ie v_1, v_2, -- v_n respectively such that $v_1 = r_1\omega$, $v_2 = r_2\omega$, - - - ; $v_n = r_n\omega$.

The translational KE of 1st particle is

$$KE_1 = \frac{1}{2} m_1 v_1^2 = \frac{1}{2} m_1 r_1^2 \omega^2$$

For 2nd particle $KE_2 = \frac{1}{2} m_2 v_2^2 = \frac{1}{2} m_2 r_2^2 \omega^2$ similarly for nth particle $KE_n = \frac{1}{2} m_n v_n^2 = \frac{1}{2} m_n r_n^2 \omega^2$

The rotational KE of a body is the sum of individual ~~kea~~ translational KE.

$$\therefore \text{Rotational KE} = \frac{1}{2} m_1 r_1^2 \omega^2 + \frac{1}{2} m_2 r_2^2 \omega^2 + \text{------} + \frac{1}{2} m_n r_n^2 \omega^2$$

$$= \frac{1}{2} (m_1 r_1^2 + m_2 r_2^2 + \text{---} + m_n r_n^2) \omega^2.$$

$$= \frac{1}{2} \sum_{i=1}^{n} m_i r_i^2 \cdot \omega^2$$

$$\boxed{\text{Rotational KE} = \frac{1}{2} I \omega^2} \text{ ----- } \left(\sum_{i=1}^{n} m_i r_i^2 = I \rightarrow \text{M-I.} \right)$$

The Rotational KE $\left(\frac{1}{2} I \omega^2\right)$ is analogous to translational KE $\left(\frac{1}{2} m v^2\right)$

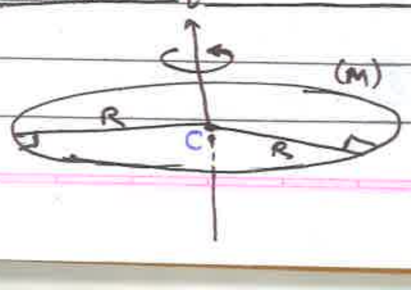
It shows that I is replace by m & ω is replace by v. So Moment of inertia is an analogous quantity for mass & so called as rotational inertia because mass represents inertia of a body in translational motion.

Moment of Inertia :- It is defined as the sum of product of mass & square of its ⊥er distance from axis of rotation of each particle of a rigid body rotating about an axis passing through it.

Moment of inertial depends upon

1) Individual masses of the particles of body.
2) Distribution of these masses about the given axis of rotation.

* Moment of Inertia of a Uniform ring:



A uniform circular ring of mass M & radius R is rotating about an its own axis (⊥er to plane & passing through centre)

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Entire mass M of ring is at dist^n R from the axis.
$\therefore$ MI of ring is given by

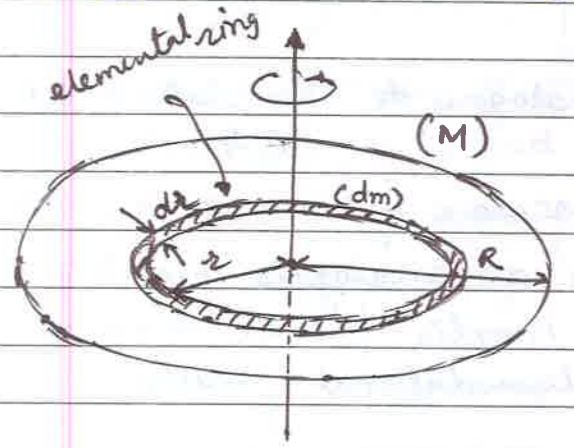
$$\boxed{I = MR^2}$$

* Moment of Inertia of a Uniform Disc

Disc is two dimensional circular object of negligible thickness. The disc is uniform, equal mass distribute per unit area, its surface density is given by

$$\sigma = \frac{m}{A} \quad \text{--- (1)}$$

Consider a uniform disc of mass M & radius R rotating about its own axis ie $\perp$er to its plane & passing through centre.



$$\therefore \sigma = \frac{M}{\pi R^2} \quad \text{--- (2)}$$

Let the disc be divided into no. of concentric rings of increasing radii from 0 to R. Consider such a ring of mass dm & radius r & thickness dr.

So area of ring is $A = 2\pi r dr$, surface density of ring is

$$\sigma = \frac{dm}{2\pi r dr} \quad \text{--- (3)}$$

$$\therefore dm = 2\pi r dr\, \sigma \quad \text{--- (4)}$$

The M.I. of ring is given by

$$I_r = dm\, r^2 \quad \text{--- (5)}$$

The MI of disc be obtained by integrating eq^n (5) from limit $r=0$ to $r=R$

$$\therefore I = \int_0^R I_r = \int_0^R dm\, r^2 \quad \text{--- (6)}$$

put ⑤ in ⑥

$$I = \int_0^R 2\pi r\,dr \cdot b \cdot r^2$$

$$= 2\pi b \int_0^R r^3\,dr$$

$$= 2\pi b \left[\frac{r^4}{4}\right]_0^R$$

$$= 2\pi b \cdot \frac{R^4}{4} \quad \text{—— ⑦}$$

put ② in ⑦

$$\therefore I = 2\pi \cdot \frac{M}{\pi R^2} \cdot \frac{R^4}{4}$$

$\boxed{I = \frac{1}{2} MR^2}$ —— ⑧ → This the MI of disc about its own axis ⊥er to its plane & passing through the centre.

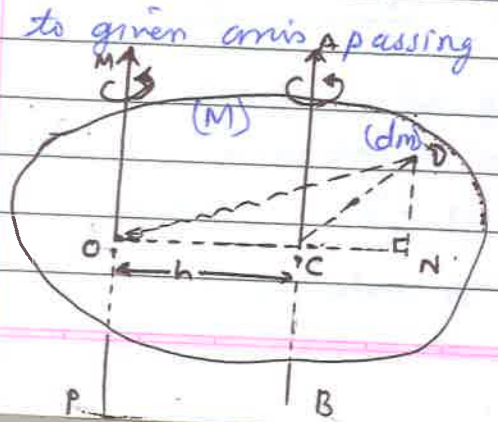
* **Radius of gyration (K):** Radius of gyration of a rigid body rotating about an axis passing through it is defined as the perpendicular distance betn the axis of rotation & at point at which whole mass of a body is suppose to be concentrated (centre of mass).

$$I = MK^2 \quad \text{—— } (K = \text{radius of gyration})$$

Larger value of K shows the mass is away from the axis & smaller value of K shows the mass is nearer to the axis of rotation.

Q- State & Explain the theorem of principle of parallel axis.

→ **Statement:** The M.I. of a rigid body about any axis of rotation is equal to the sum of its MI about a parallel axis to given axis passing through its centre of mass & the product of the mass of a body and square of the distance betn two parallel axes.



$$(I_O = I_C + Mh^2)$$

Consider a rigid body of mass M. Axis MAOP is any axis passing through point O. Axis ACB is passing through centre of mass C. of a body parallel to axis MOP. 'h' is the perpendicular distⁿ betⁿ two parallel axes.

Consider a small element of mass dm at point D. Draw ⊥er DN on OC produced.

M.I. of rigid body about an axis ACB is

$$I_C = \int (DC)^2 dm \quad — (1)$$

MI of rigid body about an axis MOP is

$$I_O = \int (DO)^2 dm \quad — (2)$$

$$I_O = \int [(DN)^2 + (NO)^2] dm.$$

$$= \int [(DN)^2 + (NC)^2 + 2NC \cdot CO + (CO)^2] dm.$$

$$= \int [(DC)^2 + 2NC \cdot h + h^2] dm$$

$$= \int (DC^2) dm + 2h \int NC \cdot dm + h^2 \int dm \quad — (3)$$

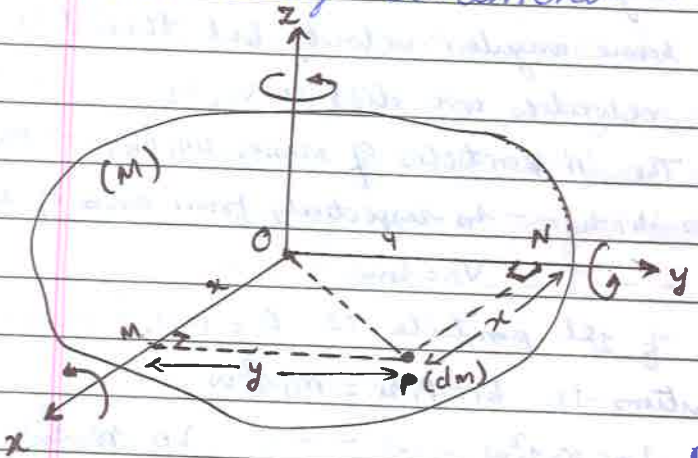
but $\int NC\, dm = 0$ as NC is the distⁿ of a point from the C.M. Any mass distribution is symmetric about the C.M.

& $h^2 \int dm = M h^2$, put (1) in (3)

$$\therefore \boxed{I_O = I_C + M h^2} \quad — (4)$$

Q. State & Explain the theorem of principle of perpendicular axes.

⟹ <u>statement:</u> The M.I of a plane laminar body about an axis perpendicular to its ~~phane~~ plane is equal to the sum of its moment of inertias about two mutually perpendicular axes & in its plane, all the three axes being concurrent.



A rigid laminar body able to rotate about three mutually perpendicular axes. x, y & z. Axes x & y are in the plane of the body while z is ⊥er to it. and all are concurrent at O. Consider a particle of mass dm at point P. Draw ⊥ers PM = y & PN = ~~x~~ on x & y axis respetively. M.I. of a body about x-axis is

$$I_x = \int y^2 dm \quad — ①$$

MI of a body about y-axis is

$$I_y = \int x^2 dm \quad — ②$$

Add ① & ②

$$I_x + I_y = \int (x^2+y^2)\, dm \quad — ③$$

MI of a body about z-axis is

$$I_z = \int (OP)^2 dm \quad — ④$$

From fig $(OP)^2 = x^2 + y^2$ - - from eqⁿ ③ & ④

$$\boxed{I_z = I_x + I_y} \quad — ⑤$$

* <u>Angular momentum</u>: The quentity in rotational mechanics alalogous to linear momentum is angular momentum or moment of linear momentum

$\vec{L} = \vec{r} \times \vec{P}$ $\vec{r}$ = position vector, $\vec{p}$ = linear momentum.

∴ $L = P \times r \sin\theta$ where θ is the angle betⁿ $\vec{P}$ & $\vec{r}$.

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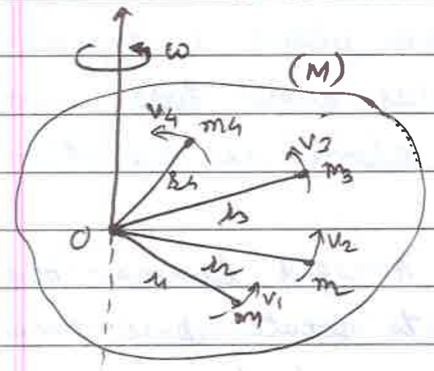
Q. Obtain an expression of angular momentum of a rigid body rotating about an axis in terms of moment of inertia.

⟹ 

A rigid body of mass M is rotating about an axis passing through it with constant angular velocity 'ω'. All particles have same angular velocity but their linear velocities are diff. ie $v_1, v_2 - - - v_n$.

The 'n' particles of masses $m_1, m_2, - - - m_n$ are situated at ⊥er distances $r_1, r_2, - - r_n$ respectively from axis of rotation.

$\therefore v_1 = r_1 w, \quad v_2 = r_2 w \quad - - - - - \quad v_n = r_n w.$

The linear momentum of 1st particle is $P_1 = mv_1 = m_1 r_1 w$.

So its angular momentum is $L_1 = P_1 r_1 = m_1 r_1^2 w$.

Similarly $L_2 = m_2 r_2^2 w, \quad L_3 = m_3 r_3^2 w \quad - - \quad - - \quad L_n = m_n r_n^2 w$.

All the angular momenta are directed along the axis of rotation.

So, the total angular momentum of rigid body is given by

$$L = m_1 r_1^2 w + m_2 r_2^2 w + - - - - + m_n r_n^2 w$$

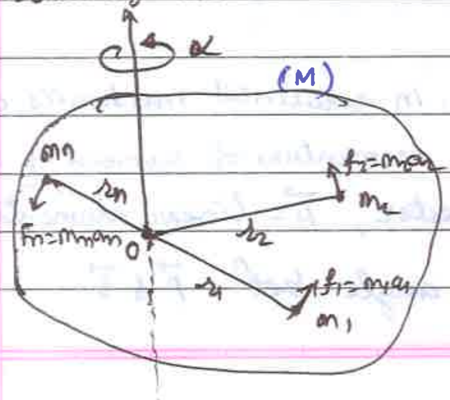
$$L = (m_1 r_1^2 + m_2 r_2^2 + - - - - - - + m_n r_n^2)\, \omega$$

$$L = \left(\sum_{i=1}^{n} m_i r_i^2\right) \cdot \omega$$

$$\therefore \boxed{L = I\omega} \quad -① $$

Eqⁿ ① is analogous to the eqⁿ of linear momentum $P = mV$. As moment of inertia I is replaces mass, which is its physical significance.

Q. Obtain an expression of torque acting on a rigid body rotating about an axis passing through it in terms of M.I.

→ 

A rigid body of mass M rotating about an axis passing through it with constant angular accⁿ α, It consist of n no. of particles of masses $m_1, m_2 - - - m_n$, situated at ⊥er distances

r_1, r_2, r_3 --- r_n respectively from axis of rotation. The angular accⁿ α is same for all particles but their linear accⁿˢ are diff. i.e a_1, a_2, a_3 - - - - a_n.

$\therefore a_1 = r_1\alpha, \; a_2 = r_2\alpha, \; - - - - - a_n = r_n\alpha.$

The force experienced by the 1st particle is $f_1 = m_1 a_1 = m_1 r_1 \alpha$ & the torque experienced by this particle is $\tau_1 = f_1 r_1 = m_1 r_1^2 \alpha$

Similarly $\tau_2 = m_2 r_2^2 \alpha, \; \tau_3 = m_3 r_3^2 \alpha$ - - - - -, $\tau_n = m_n r_n^2 \alpha$.

The directions of all torques are same. So the resultant torque acting on a rigid body is

$$\tau = \tau_1 + \tau_2 + - - - + \tau_n$$

$$\therefore \tau = m_1 r_1^2 \alpha + m_2 r_2^2 \alpha + - - - - + m_n r_n^2 \alpha$$

$$\tau = (m_1 r_1^2 + m_2 r_2^2 + - - - - + m_n r_n^2)\alpha$$

$$\tau = \left(\sum_{i=1}^{n} m_i r_i^2\right) \cdot \alpha$$

$$\therefore \boxed{\tau = I\alpha} \quad \text{--- (1)}$$

Eqⁿ is (1) is analogous to $f = ma$ for the translational motion. As the moment of inertia I replaces mass, which is the physical significance of M.I.

Q. state & prove the ~~law~~ Principle of conservation of angular momentum.

→ Statement: The angular momentum of a rigid body rotating about its axis remains constant in the absence of external unbalance torque.

The angular momentum of a rigid body is given by

$$\vec{L} = \vec{r} \times \vec{P} \quad \text{--- (1)}$$

Differentiate w.r.t. time

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}(\vec{r} \times \vec{P}) = \vec{r} \times \frac{d\vec{P}}{dt} + \frac{d\vec{r}}{dt} \times \vec{P} \quad \text{--- (2)}$$

but $\frac{d\vec{r}}{dt} = \vec{v}$ & $\vec{P} = m\vec{v}$, $\frac{d\vec{P}}{dt} = \vec{f}$

$$\therefore \frac{d\vec{L}}{dt} = \vec{r} \times \vec{F} + m(\vec{v} \times \vec{v}) \quad - - - \quad (\vec{v} \times \vec{v} = 0)$$

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$$\therefore \frac{d\vec{L}}{dt} = \vec{r} \times \vec{F} \quad \text{but} \quad \vec{\tau} = \vec{r} \times \vec{F}$$

$$\therefore \frac{d\vec{L}}{dt} = \vec{\tau}$$

If $\vec{\tau} = 0$, $\therefore \frac{d\vec{L}}{dt} = 0 \quad \therefore \vec{L} = \text{constant}$.

* **Examples of conservation of angular momentum.**

1) **Ballet Dancers**:- During ice ballet, the dancers have to undertake rounds of smaller & larger radii. The dancers come together ~~while~~ while taking the rounds of smaller radius. in this case hands & leges are close to their bodies, so M.I. becomes min^m & angular freq^n increases, to make it thrilling. while outer rounds, the dancers outstretch their legs & arms, to increase M.I. & reduces the angular speed. This is essential to prevent slipping.

ii) **Diving in a Swimming pool** (During competition):-

The divers stretch their body as to increase the M.I. Immediately after leaving the boad, they fold their body. This reduces M.I. & increases their freq^n so they can complete more rounds, in air to make the show attractive. Again while entering into water they stretch their body into a streamline shape. This allow them a smooth entry into the water.

Q. Find the eq^n of total KE of a body like sphere (wheel/cylinder) for their rolling motion.

→ Consider a body like sphere (wheel/cylinder) of mass M, radius R, & M.I. I about its own axis rolling on a horizontal plane. for pure rolling there are two motions of a body i.e circular & linear (translational) motions. as

1) Circular motion of a body as a whole about its own symmetric axis

2) Linear (Translational) motion of centre of mass at its centre

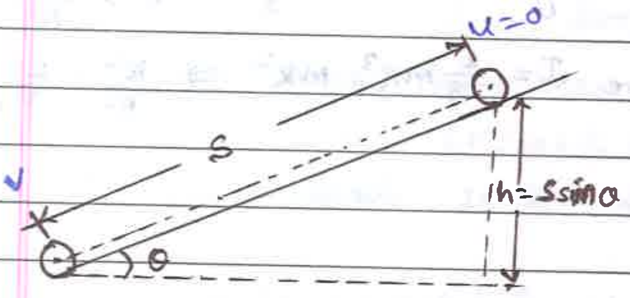
Total KE. = Translational KE + Rotational KE.

$$E = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2$$

$$= \frac{1}{2}Mv^2 + \frac{1}{2}(MK^2)\left(\frac{v^2}{R^2}\right)$$

$$\boxed{E = \frac{1}{2}Mv^2\left[1 + \frac{K^2}{R^2}\right]}$$

Q. Find the eqns of linear speed & acceleration of a rigid body while pure rolling down an inclined plane.



A rigid body like sphere of mass M, & radius R rolling down an inclined plane without slipping. The angle of inclination of plane is θ.

As the object starts rolling down, its gravitational PE is converted into KE of rolling motion. If v is the speed of C.M, I is MI of a body rolling with angular velocity ω.

E = Gravitation PE = Rolling KE $(E_{Tr} + E_{Rot})$

$$\therefore \quad Mgh = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2 \quad \text{— (1)}$$

$$Mgh = \frac{1}{2}Mv^2\left(1 + \frac{K^2}{R^2}\right)$$

$$\therefore \quad v^2 = \frac{2gh}{\left(1 + K^2/R^2\right)}$$

$$\therefore \boxed{v = \sqrt{\frac{2gh}{(1 + K^2/R^2)}}} \quad \text{— (2)}$$

For linear accn use 3rd kinematical eqn

$$v^2 = u^2 + 2as \quad \text{—— (as } u = 0)$$

$$v^2 = 2as$$

$$a = \frac{v^2}{2s} \quad \text{— (3)}$$

From fig $s = \frac{h}{\sin\theta}$ & put (1) in (3)

$$a = \frac{2gh}{\left(1+\frac{k^2}{R^2}\right)} \times \frac{1}{2} \times \frac{\sin\theta}{h}$$

$$\therefore \boxed{a = \frac{g\sin\theta}{\left(1+k^2/R^2\right)}} \quad \text{--- (4)}$$

* 1) For uniform solid sphere $I = \frac{2}{5}MR^2 = Mk^2 \Rightarrow \frac{k^2}{R^2} = \frac{2}{5}$

2) For circular ring or hollow cylinder $I = MR^2 = Mk^2 \Rightarrow \frac{k^2}{R^2} = 1.$

3) For uniform disc or solid cylinder $I = \frac{MR^2}{2} = Mk^2 \Rightarrow \frac{k^2}{R^2} = \frac{1}{2}.$

4) for a thin walled sphere $I = \frac{2}{3}MR^2 = Mk^2 \Rightarrow \frac{k^2}{R^2} = \frac{2}{3}.$